Entanglement in random tensor networks

arXiv:1601.01694, 1608.02595

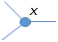
Patrick Hayden, Sepehr Nezami, Nathaniel Thomas, Xiaoliang Qi, Michael Walter, Zhao Yang

Institute for Theoretical Physics, Stanford University

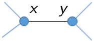
QMath13, Georgia Tech, October 2016

Tensor network states

Graphical notation:

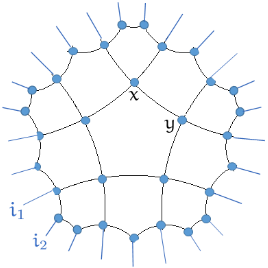


Ψ_{ijk} pure state in $(\mathbb{C}^D)^3$



$\Psi_{ijlm} = \sum_{k=1}^{D} V_{x,ijk} V_{y,klm}$ with V_x, V_y

In general: Given a graph $G = (V, E)$ and bond dimension D, define

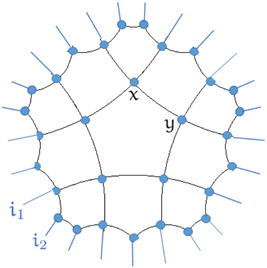


$$|\Psi\rangle = \left(\bigotimes_{\langle xy\rangle \in E} \langle xy|\right)\left(\bigotimes_{x \in V} |V_x\rangle\right)$$

- $|V_x\rangle$ tensors
- $|xy\rangle = \sum_{i=1}^{D} |i, i\rangle$ max. entangled

Model: Random tensor network states

Given a graph $G = (V, E)$ and bond dimension D, define



$$|\Psi\rangle = \left(\bigotimes_{\langle xy \rangle \in E} \langle xy| \right) \left(\bigotimes_{x \in V} |V_x\rangle \right)$$

- $|V_x\rangle$ i.i.d. random (e.g., Haar)
- $|xy\rangle = \sum_{i=1}^{D} |i, i\rangle$ max. entangled

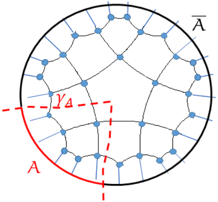
Motivation

- Mathematics: Natural 'geometric' generalization of Haar measure
- QIT: Entanglement distillation in entangled pair states

- Tensor network kinematics:

$$S(A) \leq \log(D) \min|\gamma_A|$$

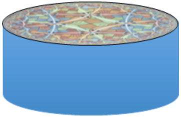
Generically saturated or fine-tuned?



- Holographic principle in quantum gravity, as realized by AdS/CFT correspondence:

$$S(A) = \frac{1}{4G_N} \min|\gamma_A|$$

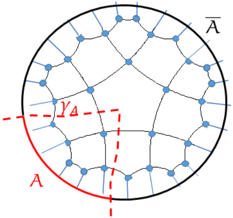
Toy models from tensor networks?



Main result I

Theorem (Bipartite entanglement)

In random tensor network states: $S(A) = \log(D) \min|\gamma_A| - O(1)$ *w.h.p.*

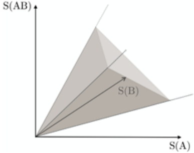


Prior work: Collins et al (random MPS), Hastings (random MERA).

Followup: Hastings (identical tensors, limiting spectral distribution in some cases).

Holographic entropy inequalities

'Holographic' entropy formula has interesting properties:



$$S(A) = c \min|\gamma_A|,$$

Can be systematically studied via entropy cone formalism of Zhang & Yeung:

- finitely many entropy inequalities (for any number of subsystems)
- combinatorial criterion for proving nonstandard entropy inequalities
- ex.: monogamy of mutual information

$$I(A:B) + I(A:C) \leq I(A:BC)$$

is unique additional inequality for fourpartite systems. But correlations are not in general monogamous – not valid for Shannon, vN entropy.

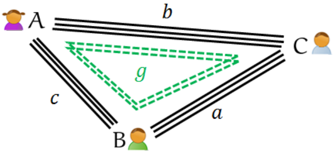
Does the mutual information in these states measure q. entanglement?

Model: Random *stabilizer* network states $D = p^n$

For this question, we use random stabilizer states as the vertex tensors $|V_x\rangle$. Then the tensor network state $|\Psi\rangle$ is also a stabilizer state. Recall:

Stabilizer state: Eigenvector of maximal abelian subgroup of Pauli group.[1]

- class of quantum states with efficient classical description
- 2-design; 3-design if and only if $p = 2$ (Küng & Gross).
- tripartite entanglement structure (Bravyi, Fattal & Gottesman):



$$I(A : B) = 2c + g$$

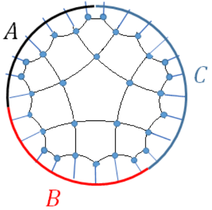
where $|\text{GHZ}\rangle \propto \sum_{j=1}^{p} |jjj\rangle$ (separable marginals).

[1] For $\mathbb{C}^p$, generated by $X|j\rangle = |j+1\rangle$, $Z|j\rangle = \exp(2\pi ij/p)|j\rangle$. For $(\mathbb{C}^p)^{\otimes n}$, use $\otimes$.

Main result II

Theorem (Tripartite entanglement)

In random *stabilizer* network states: $\#\mathrm{GHZ}(A:B:C) = O(1)$ *w.h.p.*



Prior work: Smith & Leung (single random stabilizer state).

Corollary

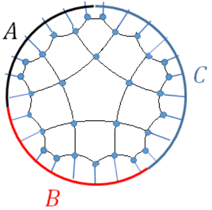
Can distill $\simeq \frac{1}{2} I(A:B)$ maximally entangled pairs by local Clifford unitaries.

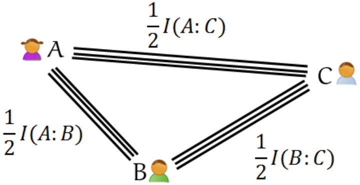
- mutual information measures entanglement *(w.h.p.)*.

Main result II

Theorem (Tripartite entanglement)

In random *stabilizer* network states: $\#\mathrm{GHZ}(A:B:C) = O(1)$ *w.h.p.*





Prior work: Smith & Leung (single random stabilizer state).

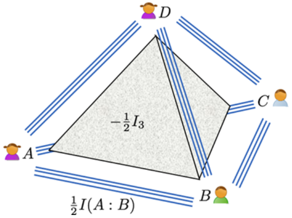
Corollary

Can distill $\simeq \frac{1}{2}I(A:B)$ maximally entangled pairs by local Clifford unitaries.

- mutual information measures entanglement *(w.h.p.)*.

Higher-partite entanglement

We can iteratively distill bipartite maximal entanglement between any two subsystems $\rightsquigarrow$ residual state has $I(A:B) = O(1)$ etc. *(w.h.p.)*



In a four-partite system, this implies *("perfect tensor")*

$$S(A), \ldots, S(D) \simeq -\frac{1}{2} I_3, \quad S(AB), \ldots, S(CD) \simeq -I_3$$

where $I_3 = I(A:B) + I(A:C) - I(A:BC)$ is the tripartite information:

- invariant under distillation, can estimate from geometry of graph
- $I_3 < 0$ diagnoses four-partite entanglement
- another proof that the mutual info is monogamous

Proof ingredient I: Spin models

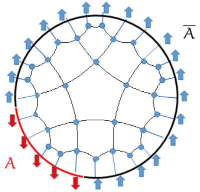
Theorem (Bipartite entanglement)

In random tensor network states: $S(A) = \log(D) \min|\gamma_A| - O(1)$ *w.h.p.*

Sketch of proof: Lower-bound Rényi entropy $S_2(A) = -\log \operatorname{tr} \rho_A^2$.

Using swap trick & second moments:

$$\mathbb{E}[\operatorname{tr} \rho_A^2] \propto Z_A = \sum_{\{s_x\}} e^{-\log D \sum_{\langle xy \rangle} (1 - s_x s_y)/2}$$



Ferromagnetic Ising model at $\beta = \log D$ with mixed boundary conditions.

- $S_2(A)$ is related to free energy $F = -\log Z_A$
- large D/low T: dominated by energy of minimal domain wall

More precise estimates possible in terms of geometry of graph!

Proof ingredient I: Spin models

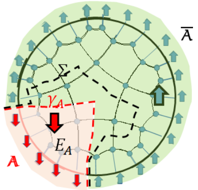
Theorem (Bipartite entanglement)

In random tensor network states: $S(A) = \log(D) \min|\gamma_A| - O(1)$ *w.h.p.*

Sketch of proof: Lower-bound Rényi entropy $S_2(A) = -\log \operatorname{tr} \rho_A^2$.

Using swap trick & second moments:

$$\mathbb{E}[\operatorname{tr} \rho_A^2] \propto Z_A = \sum_{\{s_x\}} e^{-\log D \sum_{\langle xy \rangle} (1 - s_x s_y)/2}$$



Ferromagnetic Ising model at $\beta = \log D$ with mixed boundary conditions.

- $S_2(A)$ is related to free energy $F = -\log Z_A$
- large D/low T: dominated by energy of minimal domain wall

More precise estimates possible in terms of geometry of graph!

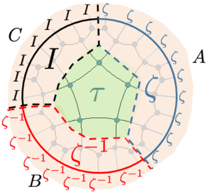
Proof ingredient II: Higher moments

Theorem (Tripartite entanglement)

In random *stabilizer* network states: $\#\mathrm{GHZ}(A\!:\!B\!:\!C) = O(1)$ *w.h.p.*

Sketch of proof: Diagnose via

$$\#\,\mathrm{GHZ} = S(A) + S(B) + S(C) - \log \operatorname{tr}(\rho_{AB}^{T_B})^3$$



- ferromagnetic spin model with variables $\pi_x \in S_3$, cyclic boundary conditions

Lemma (Third moment of random stabilizer state, $p \equiv 2 \pmod 3$)

$$\mathbb{E}[\psi^{\otimes 3}] = \frac{1}{D(D+1)(D+p)} \sum_{T \in G_3(p)} r(T)^{\otimes n}$$

with $G_3(p)$ the group of orthogonal & stochastic 3×3-matrices over $\mathbb{F}_p$.

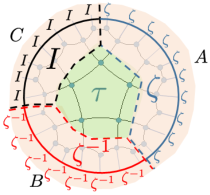
Proof ingredient II: Higher moments $D = p^n$

Theorem (Tripartite entanglement)

In random *stabilizer* network states: $\#\mathrm{GHZ}(A:B:C) = O(1)$ *w.h.p.*

Sketch of proof: Diagnose via

$$\#\,\mathrm{GHZ} = S(A) + S(B) + S(C) - \log \mathrm{tr}(\rho_{AB}^{T_B})^3$$



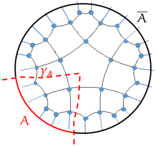
- ferromagnetic spin model with variables $\pi_x \in G_3(p)$, cyclic boundary conditions

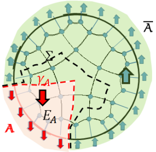
Lemma (Third moment of random stabilizer state, $p \equiv 2 \pmod 3$)

$$\mathbb{E}[\psi^{\otimes 3}] = \frac{1}{D(D+1)(D+p)} \sum_{T \in G_3(p)} r(T)^{\otimes n}$$

with $G_3(p)$ the group of orthogonal & stochastic 3×3-matrices over $\mathbb{F}_p$.

Summary and outlook





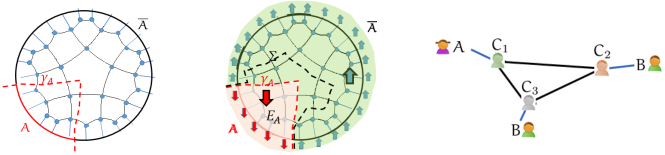
Random tensor networks:

- Bipartite & multipartite entanglement properties dictated by geometry
- Techniques: spin models for random tensor averages, moments

What we did *not* discuss today:

- Connection to entanglement distillation
- Geometric subsystem codes ('holographic' codes of Pastawski *et al*)
- Toy model & explanation of some structural features of AdS/CFT

Thank you for your attention!



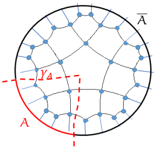
Random tensor networks:

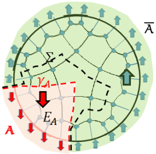
- Bipartite & multipartite entanglement properties dictated by geometry
- Techniques: spin models for random tensor averages, moments

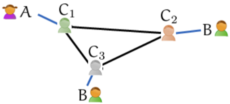
What we did *not* discuss today:

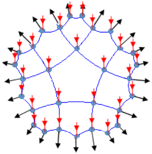
- Connection to entanglement distillation
- Geometric subsystem codes (‘holographic’ codes of Pastawski *et al*)
- Toy model & explanation of some structural features of AdS/CFT

Thank you for your attention!









Random tensor networks:

- Bipartite & multipartite entanglement properties dictated by geometry
- Techniques: spin models for random tensor averages, moments

What we did *not* discuss today:

- Connection to entanglement distillation
- Geometric subsystem codes ('holographic' codes of Pastawski *et al*)
- Toy model & explanation of some structural features of AdS/CFT

Thank you for your attention!